



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

# NOTE ON STOKES'S THEOREM IN CURVILINEAR CO-ORDINATES.

BY ARTHUR GORDON WEBSTER.

Presented April 13, 1898.

THE expressions for the components of the curl of a vector point-function, when required in terms of orthogonal curvilinear co-ordinates, are usually obtained by direct transformation of their values in rectangular co-ordinates.

The proof of Stokes's Theorem, given in my Lectures on Electricity and Magnetism, due to Helmholtz, may be easily adapted to curvilinear co-ordinates so as to prove the theorem independently of rectangular co-ordinates.

Let  $P_1, P_2, P_3$ , be the projections of a vector  $P$  on the varying directions of the co-ordinate axes at any point. Let the projections on the same axes of the arc  $ds$  of a curve connecting the points  $A$  and  $B$  be  $ds_1, ds_2, ds_3$ . The theorem concerns the line integral of the resolved component of the vector along the given curve.

$$\begin{aligned} I &= \int_A^B P \cos (P, ds) ds \\ &= \int_A^B P_1 ds_1 + P_2 ds_2 + P_3 ds_3. \end{aligned}$$

But in terms of the curvilinear co-ordinates  $\rho_1, \rho_2, \rho_3$ , we have

$$ds_1 = \frac{d\rho_1}{h_1}, \quad ds_2 = \frac{d\rho_2}{h_2}, \quad ds_3 = \frac{d\rho_3}{h_3},$$

where

$$h_s^2 = \left( \frac{\partial \rho_s}{\partial x} \right)^2 + \left( \frac{\partial \rho_s}{\partial y} \right)^2 + \left( \frac{\partial \rho_s}{\partial z} \right)^2. \quad s = 1, 2, 3.$$

Let us now make an infinitesimal transformation of the curve, so that the transformed curve shall lie on a given surface containing  $A$  and  $B$ ,

and shall itself pass through those points. Then the change in the integral due to changes in the co-ordinates  $\delta \rho_1, \delta \rho_2, \delta \rho_3$ , is,

$$\begin{aligned} \delta I &= \delta \int \frac{P_1}{h_1} d\rho_1 + \frac{P_2}{h_2} d\rho_2 + \frac{P_3}{h_3} d\rho_3 \\ &= \int \delta \left( \frac{P_1}{h_1} \right) d\rho_1 + \delta \left( \frac{P_2}{h_2} \right) d\rho_2 + \delta \left( \frac{P_3}{h_3} \right) d\rho_3 + \frac{P_1}{h_1} d\delta \rho_1 \\ &\quad + \frac{P_2}{h_2} d\delta \rho_2 + \frac{P_3}{h_3} d\delta \rho_3. \end{aligned}$$

The last three terms may be integrated by parts, giving

$$\int_A^B \frac{P_s}{h_s} d\delta \rho_s = \frac{P_s}{h_s} \delta \rho_s \Big|_A^B - \int_A^B \delta \rho_s d \left( \frac{P_s}{h_s} \right),$$

and the integrated part vanishing at the limits,

$$\begin{aligned} \delta I &= \int \delta \left( \frac{P_1}{h_1} \right) d\rho_1 + \delta \left( \frac{P_2}{h_2} \right) d\rho_2 + \delta \left( \frac{P_3}{h_3} \right) d\rho_3 - \delta \rho_1 d \left( \frac{P_1}{h_1} \right) \\ &\quad - \delta \rho_2 d \left( \frac{P_2}{h_2} \right) - \delta \rho_3 d \left( \frac{P_3}{h_3} \right). \end{aligned}$$

Performing the operations denoted by  $\delta$  and  $d$ , and collecting the terms which do not cancel,

$$\begin{aligned} \delta I &= \int \left[ \left( \delta \rho_2 d\rho_3 - \delta \rho_3 d\rho_2 \right) \left\{ \frac{\delta}{\delta \rho_2} \left( \frac{P_3}{h_3} \right) - \frac{\delta}{\delta \rho_3} \left( \frac{P_2}{h_2} \right) \right\} \right. \\ &\quad + \left( \delta \rho_3 d\rho_1 - \delta \rho_1 d\rho_3 \right) \left\{ \frac{\delta}{\delta \rho_3} \left( \frac{P_1}{h_1} \right) - \frac{\delta}{\delta \rho_1} \left( \frac{P_3}{h_3} \right) \right\} \\ &\quad \left. + \left( \delta \rho_1 d\rho_2 - \delta \rho_2 d\rho_1 \right) \left\{ \frac{\delta}{\delta \rho_1} \left( \frac{P_2}{h_2} \right) - \frac{\delta}{\delta \rho_2} \left( \frac{P_1}{h_1} \right) \right\} \right]. \end{aligned}$$

Now the changes  $\delta \rho_s, d\rho_s$ , in the co-ordinates correspond to distances  $\frac{\delta \rho_s}{h_s}, \frac{d\rho_s}{h_s}$ , measured along the co-ordinate lines, and the determinant of these distances,

$$\frac{1}{h_2 h_3} (\delta \rho_2 d\rho_3 - \delta \rho_3 d\rho_2),$$

is equal to the area of the projection on the surface  $\rho_1$  of the infinitesimal parallelogram swept over by the arc  $ds$  during the transformation. Calling this area  $dS$ , and its normal  $n$ , we have

$$\frac{1}{h_2 h_3} (\delta \rho_2 d\rho_3 - \delta \rho_3 d\rho_2) = \cos (nn_1) dS,$$

$$\delta \rho_2 d\rho_3 - \delta \rho_3 d\rho_2 = h_2 h_3 \cos (nn_1) dS.$$

Now, repeating the transformation so that the original curve 1 passes into a second given curve 2, the total change is represented by the surface integral over the surface lying between the curves,

$$\begin{aligned} \int \delta I = I_2 - I_1 = \iint & \left[ h_2 h_3 \left\{ \frac{\delta}{\delta \rho_2} \left( \frac{P_3}{h_3} \right) - \frac{\delta}{\delta \rho_3} \left( \frac{P_2}{h_2} \right) \right\} \cos (nn_1) \right. \\ & + h_3 h_1 \left\{ \frac{\delta}{\delta \rho_3} \left( \frac{P_1}{h_1} \right) - \frac{\delta}{\delta \rho_1} \left( \frac{P_3}{h_3} \right) \right\} \cos (nn_2) \\ & \left. + h_1 h_2 \left\{ \frac{\delta}{\delta \rho_1} \left( \frac{P_2}{h_2} \right) - \frac{\delta}{\delta \rho_2} \left( \frac{P_1}{h_1} \right) \right\} \cos (nn_3) \right] dS. \end{aligned}$$

But the difference of the line integrals  $I_2 - I_1$  is the line integral around the closed contour 12, so that we have the line integral of the tangential component of the vector  $P$  around the closed contour proved equal to the surface integral over a surface bounded by the contour of the normal component of a vector  $\Omega$  whose components are

$$\begin{aligned} \omega_1 &= h_2 h_3 \left\{ \frac{\delta}{\delta \rho_2} \left( \frac{P_3}{h_3} \right) - \frac{\delta}{\delta \rho_3} \left( \frac{P_2}{h_2} \right) \right\} \\ \omega_2 &= h_3 h_1 \left\{ \frac{\delta}{\delta \rho_3} \left( \frac{P_1}{h_1} \right) - \frac{\delta}{\delta \rho_1} \left( \frac{P_3}{h_3} \right) \right\} \\ \omega_3 &= h_1 h_2 \left\{ \frac{\delta}{\delta \rho_1} \left( \frac{P_2}{h_2} \right) - \frac{\delta}{\delta \rho_2} \left( \frac{P_1}{h_1} \right) \right\}. \end{aligned}$$

The vector  $\Omega$  is called the curl of  $P$ .